• Adrian - Part 2

• 2nd order linear elliptic:

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right) \nabla_x u_\varepsilon(x) \cdot \nabla_x \phi(x)\, dx = \int_\Omega f(x)\, \Phi(x)\, dx$$

$u_\varepsilon \in H_0^1(\Omega)$.

$\rightsquigarrow$ by standard machinery, $(u_\varepsilon)_\varepsilon$ bdd in H_0^1

$\Downarrow$ compactness

- $u_\varepsilon \rightharpoonup u_0$ in H^1
- $u_\varepsilon \rightarrow u_0$ in L^2
- $u_\varepsilon \xrightarrow{2-S} u_0$
- $\nabla u_\varepsilon \xrightarrow{2-S} \nabla_x u_0 + \nabla_y u_1$

$\rightarrow$ consider the test functions:

$$\Phi(x) = \Phi_0(x) + \varepsilon\, \Phi_1\left(x, \frac{x}{\varepsilon}\right)$$

$$\rightsquigarrow \nabla_x \Phi = \nabla_x \Phi_0 + \varepsilon \nabla_x \Phi_1\left(x, \frac{x}{\varepsilon}\right) + \nabla_y \Phi_1\left(x, \frac{x}{\varepsilon}\right)$$

we have:

$$\int_\Omega \Big[A\left(\frac{x}{\varepsilon}\right) \nabla_x u_\varepsilon(x) \cdot \nabla_x \Phi_0(x) + A\left(\frac{x}{\varepsilon}\right) \nabla_x u_\varepsilon(x) \cdot \varepsilon \nabla_x \Phi_1\left(x, \frac{x}{\varepsilon}\right)$$

$$+ A\left(\frac{x}{\varepsilon}\right) \nabla_y \Phi_1\left(x, \frac{x}{\varepsilon}\right) \Big] dx = \int_\Omega f \Phi \, dx$$

$$\downarrow$$

$$\int_\Omega f \Phi_0 \, dx$$

Let us consider term by term on the lhs:

- $$\int_\Omega \nabla_x u_\varepsilon(x) A^T\left(\frac{x}{\varepsilon}\right) \nabla_x \Phi_0(x) \, dx \longrightarrow$$

$$\longrightarrow \int_{\Omega \times Y} \left(\nabla_x u_0(x) + \nabla_y u_1(x,y)\right) A^T(y) \nabla_x \Phi_0(x) \, dx\, dy$$

- $$\int_\Omega \nabla_x u_\varepsilon(x) A^T\left(\frac{x}{\varepsilon}\right) \nabla_y \Phi_1\left(x, \frac{x}{\varepsilon}\right) dx$$

$$\longrightarrow \int_{\Omega \times Y} \left(\nabla_x u_0(x) + \nabla_y u_1(x,y)\right) A^T(y) \nabla_y \Phi_1(x,y) \, dx\, dy$$

- the other one goes to 0.

We thus obtain the weak formulation:

$$\int_{\Omega\times Y} A(y)\left(\nabla_x u_0(x) + \nabla_y u_1(x,y)\right)\left(\nabla_x \Phi_0(x) + \nabla_y \Phi_1(x,y)\right) dx\,dy$$

$$= \int_{\Omega\times Y} f(x)\,\Phi_0(x)\,dx\,dy$$ [Y is the unit cube]

$\rightsquigarrow$ the new formulation is symmetric in $u_0 \rightsquigarrow \Phi_0$, $u_1 \rightsquigarrow \Phi_1$

$\Downarrow$

we can apply Lax-Milgram to get the existence of a solution

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega, H^1_{per}).$$

Moreover, we can recover the cell & the homogenized pb by a choice of a test functions:

- $\Phi_0 = 0$; $\int_{\Omega\times Y} A(y)\left(\nabla_x u_0(x) + \nabla_y u_1(x,y)\right) \cdot \nabla_y \Phi_1(x,y)\,dx\,dy = 0$

- $\Phi_1(x,y) = \theta(x)\psi(y)$:

$$\int_{\Omega\times Y} A(y)\left(\nabla_x u_0(x) + \nabla_y u_1(x,y)\right) \nabla_y \psi\, \theta(x)\,dx\,dy = 0$$

$\Longrightarrow$ <u>cell-pb</u>

$\rightsquigarrow$ similarly, if $\Phi_1 = 0 \rightsquigarrow$ homogenized pb

- 2-scale convergence & Γ-convergence:

- $$J_\varepsilon(u) := \frac{1}{2}\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla_x u \cdot \nabla_x u \, dx - \int_\Omega f u \, dx$$

Minimizers of J_ε correspond to solutions of our PDE.

- idea: $$J_\varepsilon \xrightarrow{\Gamma} J_0(u) := \int_\Omega A_{hom} \nabla u \cdot \nabla u - \int_\Omega f u$$

with respect to the L^2-metric

i) Γ-liminf: assume $u_\varepsilon \xrightarrow{L^2} u$, and $\underline{\lim}_\varepsilon J_\varepsilon(u_\varepsilon) < +\infty$.

$\Rightarrow$ $(u_\varepsilon)_\varepsilon$ bdd in $H^1_0(\Omega)$

$\Rightarrow$ 2-scale compactness: $\nabla u^\varepsilon \xrightarrow{2-s} \nabla_x u_0 + \nabla_y u_1$, where u_1 doesn't necessarily solve the cell pb.
we want to show that we do better if u_1 solves the cell pb.

[forget about the continuous term $\int f u$]

$$J_\varepsilon(u_\varepsilon) = \int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x) \nabla u_\varepsilon(x) =$$

because we are not sure about the regularity of $\nabla_y u_1$

$$= \int_\Omega A\left(\frac{x}{\varepsilon}\right)\left(\nabla u_\varepsilon - \nabla_x u^0 - \nabla_y \varphi\left(x, \frac{x}{\varepsilon}\right)\right) \cdot \left(\nabla u_\varepsilon - \nabla_x u^0 - \nabla_y \varphi\left(x, \frac{x}{\varepsilon}\right)\right)$$

$$+ 2\int_\Omega A\left(\frac{x}{\varepsilon}\right)\left(\nabla u_\varepsilon \cdot \left(\nabla_x u_0 + \nabla_y \varphi\left(x, \frac{x}{\varepsilon}\right)\right)\right)$$

$$- \int_\Omega A\left(\frac{x}{\varepsilon}\right)\left(\nabla_x u^0 + \nabla_y \varphi\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla_x u^0 + \nabla_y \varphi\right)$$

take the lim:

we forget the first term

$$\lim J_\varepsilon(u_\varepsilon) \geq 2\int_{\Omega\times Y} A(y)(\nabla_x u_0 + \nabla_y u_1)(\nabla_x u_0 + \nabla_y \varphi_1)\,dx\,dy$$

$$\underbrace{- \int_{\Omega\times Y} A(y)(\nabla_x u^0 + \nabla_y \varphi_1)(\nabla_x u^0 + \nabla_y \varphi_1)\,dx\,dy}_{\text{by Riemann-Lebesgue lemma}}$$

⇝ by density $\varphi_1 \to u_1$

$$= \int_{\Omega\times Y} A(y)(\nabla_x u_0 + \nabla_y u_1)(\nabla_x u^0 + \nabla_y u^1)$$

$$\geq \int_{\Omega\times Y} A(y)(\nabla_x u_0 + \nabla_y \tilde{u}_1)(\nabla_x u_0 + \nabla_y \tilde{u}_1).$$

by the cell pb: let $\tilde{u}_1$ be the solution of the cell pb associated with u_0

ii) Γ-limsup: fix $u_0 \in H^1_0$; we want to find $(u_\varepsilon)_\varepsilon$ s.t. • $u_\varepsilon \xrightarrow{L^2} u_0$

• $J(u_0) \geq \limsup J_\varepsilon(u_\varepsilon)$

$$u_\varepsilon(x) = u_0(x) + \varepsilon \underbrace{u_1\left(x, \frac{x}{\varepsilon}\right)}_{\text{solution of the cell pb}}$$

⇝ better, φ_1 s.t. $\varphi_1 \sim u_1$ in L^2